DTP/99/90
July 1999

Complete Renormalization Group Improvement - Avoiding Scale Dependence in QCD Predictions

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Abstract

We show that dimensionful renormalization scheme parameters such as the renormalization or factorization scale can be completely eliminated from perturbative QCD predictions provided that all the ultraviolet logarithms involving the physical energy scale Q are completely resummed.

The problem of the renormalization scale (scheme) dependence of fixed-order perturbative QCD predictions continues to frustrate attempts to make reliable determinations of the underlying dimensional transmutation parameter of the theory , Λ_{QCD} (usually $\Lambda_{\overline{MS}}$ or $\alpha_s(M_Z)$ are the fitted quantities). Whilst a number of proposals for controlling or avoiding this difficulty have been advanced [1, 2, 3, 4] no consensus has been reached, with the result that in experimental fits attempts are made to estimate an ad hoc "renormalization scale" uncertainty [5]. This "theoretical error" can be larger than the actual experimental errors, and in our view can potentially mislead as to both the central value of $\Lambda_{\overline{MS}}$ and the likely importance of uncalculated higher-order corrections [4, 6].

It is undeniable that with standard fixed-order renormalization group (RG-) improvement there *is* a scheme dependence problem. Any proposed "solution" must, therefore , amount to special pleading for a particular choice of scheme, motivated by albeit reasonable considerations imported from outside perturbative field theory. Examples include the BLM approach of Brodsky and collaborators [3] in which the piece of the next-to-leading coefficient proportional to the first beta-function coefficient is absorbed into the coupling, motivated by various QED examples; or the Principle of Minimal Sensitivity (PMS) criterion of Stevenson [1] , where the scheme is to be chosen such that the perturbative approximation is as insensitive as possible to changes in scheme. In this paper we wish to emphasise that the *renormalization scale* dependence of fixed-order QCD perturbation theory is due to the incomplete nature of the standard RG-improvement carried out. The idea will be that the dependence of a dimensionless QCD observable $\mathcal{R}(Q)$ on the dimensionful parameter Q (e.g. the c.m. energy in e^+e^- annihilation) is obviously completely independent of how the theory is renormalized. The perturbative coefficients and the coupling $\alpha_s(\mu)$ in contrast manifestly depend on the dimensionful renormalization scale μ , via the presence of logarithms $\ln(\mu/\tilde{\Lambda})$, where $\tilde{\Lambda}$ is universal and depends on the subtraction procedure used to absorb infinities (usually $\overline{MS}$). The Q-dependence, however , is built by ultraviolet (UV) logarithms $\ln(Q/\Lambda_{\mathcal{R}})$ with $\Lambda_{\mathcal{R}}$ completely independent of the renormalization procedure. The asymptotic Q-dependence of $\mathcal{R}(Q)$ is determined by the dimensionful parameter $\Lambda_{\mathcal{R}}$ which is a physical property of the observable, independent of the renormalization scheme [4]. The key observation is that one should keep μ *independent* of Q. If this is done standard

fixed-order RG-improved predictions are self-evidently inadequate since they do not satisfy asymptotic freedom , $\mathcal{R}(Q)\to 0$ as $Q\to\infty$. This property only results if *all* the RG-predictable $\ln(Q/\Lambda_{\mathcal{R}})$ terms are resummed to all-orders building a $1/\ln(Q/\Lambda_{\mathcal{R}})$ behaviour. This summation may be accomplished with any choice of μ , but in doing so all μ-dependence cancels between the $\ln(\mu/\tilde{\Lambda})$ and $\alpha_s(\mu)$ terms. Thus the complete resummation of all the RG-predictable UV logarithms gives μ-independent predictions. One has traded unphysical μ-dependence for the correct physical Q-dependence.

In the standard RG-improvement one truncates the resummation of the $\ln(Q/\Lambda_{\mathcal{R}})$ terms and uses a Q-dependent scale $\mu = xQ$, this "incomplete" improvement yields an x-dependent result. Clearly the UV logarithms are physical and leaving out an infinite subset of them gives the resulting fixed-order approximation an unphysical scale dependence. Of course there is still residual scheme dependence since fixed-order predictions will depend on the other dimensionless parameters specifying the scheme, which can be taken to be the non-universal beta-function coefficients [1] , but dimensionful scheme dependence parameters can always be eliminated by complete resummation of all Q-dependent logarithms.

There are close links with the Effective Charge approach of Grunberg [2] which focusses on building the Q-dependence of $\mathcal{R}(Q)$. Incomplete improvement with μ chosen to be the effective charge (or fastest apparent convergence (FAC)) scale is equivalent to the "complete" RG-improvement outlined above. Similar remarks apply to examples such as moments of leptoproduction structure functions where there are two (or more) dimensionful scales, for instance the renormalization scale μ and in addition a factorization scale M. In this case one has $\ln(M/\tilde{\Lambda})$, $\ln(\mu/\tilde{\Lambda})$ logarithms as well as "physical" UV logarithms involving Q, which are independent of the renormalization and factorization conventions. Again all μ and M dependence is eliminated provided that *all* the physical UV logarithms are resummed [7].

We begin by briefly reviewing the problem of parametrizing RS-dependence, and define the concept of RG-predictable terms. Consider the dimensionless QCD observable $\mathcal{R}(Q)$, dependent on the single energy scale Q (we assume massless quarks). Without loss of generality, by raising to a power and scal-

ing, we can arrange that $\mathcal{R}(Q)$ has a perturbation series of the form,

$$\mathcal{R}(Q) = a + r_1 a^2 + r_2 a^3 + \ldots + r_n a^{n+1} + \ldots \ , \tag{1}$$

where $a \equiv \alpha_s(\mu)/\pi$ is the RG-improved coupling. The μ-dependence of a is governed by the beta-function equation,

$$\frac{\partial a}{\partial \ln \mu} = -b a^2 (1 + ca + c_2 a^2 + \ldots + c_n a^n + \ldots) \ . \tag{2}$$

Here b and c are the first two universal terms of the QCD beta-function

$$b = \frac{33 - 2N_f}{6} \ , \tag{3}$$

$$c = \frac{153 - 19N_f}{12b} \ , \tag{4}$$

with N_f the number of active quark flavours. As demonstrated by Stevenson [1] the renormalization scheme may be completely labelled by the variables $\tau \equiv b \ln(\mu/\tilde{\Lambda})$ and the non-universal beta-function coefficients $c_2, c_3, \ldots$. $a(\tau, c_2, c_3, \ldots)$ is obtained as the solution of the transcendental equation

$$\frac{1}{a} + c \ln\left(\frac{ca}{1+ca}\right) = \tau - \int_0^a dx \ \left(-\frac{1}{B(x)} + \frac{1}{x^2(1+cx)}\right) \ , \tag{5}$$

where $B(x) \equiv x^2(1 + cx + c_2 x^2 + c_3 x^3 + \ldots)$. This is obtained by integrating Eq.(2) with a suitable choice of boundary condition related to the definition of $\tilde{\Lambda}$.

For our purposes it will be useful to label the RS using r_1, the next-to-leading order (NLO) perturbative coefficient, rather than τ. This is possible because [1]

$$\tau - r_1 = \rho_0(Q) \equiv b \ln(Q/\Lambda_{\mathcal{R}}) \ , \tag{6}$$

where ρ_0 is an RS-invariant, hence τ can be traded for r_1. $\Lambda_{\mathcal{R}}$ is a dimensionful scale dependent on the particular observable. It is related to the *universal* dimensional transmutation parameter $\tilde{\Lambda}_{\overline{MS}}$ by

$$\Lambda_{\mathcal{R}} \equiv e^{r/b} \tilde{\Lambda}_{\overline{MS}} \ , \tag{7}$$

where $r \equiv r_1^{\overline{MS}}(\mu = Q)$. The righthand side of Eq.(7) is independent of the subtraction scheme employed, and as advertised $\Lambda_{\mathcal{R}}$ has a physical significance, being directly related to the asymptotic Q-dependence of $\mathcal{R}(Q)$ [2, 4, 6]. Equations (5),(6) can then be used to define $a(r_1, c_2, c_3, \ldots)$.

We next turn to the RS-dependence of the perturbative coefficients r_i. This must be such as to cancel the RS-dependence of 'a' when the series is summed to all-orders. The self-consistency of perturbation theory [1] demands that the result of a N^{n}LO calculation (terms up to and including $r_n a^{n+1}$) in two *different* schemes should differ by O(a^{n+2}). This implies the following dependences of the r_i on the scheme parameters- $r_2(r_1, c_2)$, $r_3(r_1, c_2, c_3)$,..., $r_n(r_1, c_2, c_3, \ldots, c_n)$. The self-consistency requirement can be used to derive expressions for the partial derivatives of the r_n with respect to the scheme parameters. For instance for r_2 one has

$$\frac{\partial r_2}{\partial r_1} = 2r_1 + c, \;\; \frac{\partial r_2}{\partial c_2} = -1, \;\; \frac{\partial r_2}{\partial c_3} = 0, \ldots \;\; . \tag{8}$$

On integrating these expressions one finds

$$\begin{aligned} r_2(r_1, c_2) &= {r_1}^2 + c r_1 + X_2 - c_2 \\ r_3(r_1, c_2, c_3) &= {r_1}^3 + \frac{5}{2} c {r_1}^2 + (3X_2 - 2c_2) r_1 + X_3 - \frac{1}{2} c_3 \\ \vdots \qquad & \qquad \vdots \, . \end{aligned} \tag{9}$$

In general the structure is

$$r_n(r_1, c_2, \ldots, c_n) = \hat{r}_n(r_1, c_2, \ldots, c_{n-1}) + X_n - c_n/(n-1) \; . \tag{10}$$

Here $\hat{r}_n$ is an n^{th} order polynomial in r_1 which is determined given a complete N^{n-1}LO calculation. X_n is a Q-independent and RS-invariant constant of integration and can only be determined given a complete N^{n}LO calculation. $\hat{r}_n$ is the "RG-predictable" part of r_n , and X_n is "RG-unpredictable". Thus, given a NNLO calculation in the $\overline{MS}$ scheme with $\mu = Q$ one can determine the RS-invariant

$$X_2 = r_2^{\overline{MS}}(\mu = Q) - \left(r_1^{\overline{MS}}(\mu = Q)\right)^2 - c \, r_1^{\overline{MS}}(\mu = Q) + c_2^{\overline{MS}} \; , \tag{11}$$

where

$$c_2^{\overline{MS}} = \frac{77139 - 15099 N_f + 325 N_f^2}{1728 b} \; . \tag{12}$$

By a complete N^{n}LO calculation we mean that $c_2, c_3, \ldots, c_n$ have been computed as well as $r_1, r_2, \ldots, r_n$. We note in passing that $c_3^{\overline{MS}}$ has now been calculated [9].

Using Eqs.(9) we can now exhibit the explicit RS-dependence of the terms of Eq.(1),

$$\mathcal{R}(Q) = a + r_1 a^2 + (r_1^2 + c r_1 + X_2 - c_2)a^3 + (r_1^3 + \frac{5}{2} c r_1^2 + (3X_2 - 2c_2) r_1 + X_3 - \frac{1}{2} c_3)a^4 + \ldots , \tag{13}$$

where $a \equiv a(r_1, c_2, c_3, \ldots)$. We adopt the principle that at any given order of Feynman diagram calculation *all* known (RG-predictable) terms should be resummed to all-orders. Given a NLO calculation r_1 is known but $X_2, X_3, \ldots$ are unknown. Thus the complete subset of known terms in Eq.(13) at NLO is

$$a_0 \equiv a + r_1 a^2 + (r_1^2 + c r_1 - c_2)a^3 + (r_1^3 + \frac{5}{2} c r_1^2 - 2 c_2 r_1 - \frac{1}{2} c_3)a^4 + \ldots . \tag{14}$$

The sum of these terms, a_0 , can be simply determined using the following two-step argument. The infinite subset of terms in Eq.(14) has an RS-independent sum, since the $X_2, X_3, \ldots$, -dependent terms cannot cancel their RS-dependence, and we know that the full sum of Eq.(13) is RS-invariant. Each term is a multinomial in $r_1, c_2, c_3, \ldots$. Using the RS-independence we can set $r_1 = 0, c_2 = 0, c_3 = 0, \ldots$, in which case all terms but the first in Eq.(14) vanish and we obtain $a_0 = a(r_1 = 0, c_2 = 0, c_3 = 0, \ldots, c_n = 0, \ldots)$. So at NLO this corresponds to working in an "'t Hooft scheme" with $c_2 = c_3 = \ldots = 0$ [10], and with $r_1 = 0$. From Eq.(6) $r_1 = 0$ corresponds to $\tau = b \ln(Q/\Lambda_{\mathcal{R}})$ or to an $\overline{MS}$ scale $\mu = e^{-r/b} Q$. This is the so-called "fastest apparent convergence" (FAC) or effective charge (EC) scale [1, 2]. From Eq.(5) we find that a_0 satisfies

$$\frac{1}{a_0} + c \ln\left(\frac{c a_0}{1 + c a_0}\right) = b \ln\left(\frac{Q}{\Lambda_{\mathcal{R}}}\right) . \tag{15}$$

We note to avoid confusion that the definition of $\tilde{\Lambda}$ on which Eq.(5) is based [1] differs from that usually used for $\Lambda_{\overline{MS}}$. In terms of the standard definition we have,

$$\Lambda_{\mathcal{R}} = e^{r/b} \left(\frac{2c}{b}\right)^{-c/b} \Lambda_{\overline{MS}} . \tag{16}$$

If a NNLO calculation has been completed , then X_2 can be determined (as in Eq.(11)), and a further infinite subset of terms are known and can be resummed to all-orders,

$$X_2 {a_0}^3 = X_2 a^3 + 3X_2 r_1 a^4 + \dots \ . \tag{17}$$

The RS-independence of the sum and the multinomial structure of the coefficients again leads to a resummed result involving a_0. We finally arrive at

$$\mathcal{R}(Q) = a_0 + X_2 {a_0}^3 + X_3 {a_0}^4 + \ldots + X_n {a_0}^{n+1} + \ldots \ , \tag{18}$$

which is simply the perturbation series in the RS with $r_1 = c_2 = c_3 = \ldots = c_n = \ldots = 0$. As is obvious from Eqs.(9), $X_n = r_n(r_1 = 0, c_2 = 0, \ldots, c_n = 0)$.

Unfortunately the result obtained by resumming all RG-predictable terms depends on our choice of $r_1, c_2, c_3, \ldots, c_n, \ldots$ as the parameters used to label the scheme. Whilst this choice is natural and straightforward it is evidently not unique. We could equally consider a translated set of parameters-: $\tilde{r}_1 = r_1 - \overline{r}_1,\ \tilde{c}_2 = c_2 - \overline{c}_2,\ \ldots, \tilde{c}_n = c_n - \overline{c}_n$, where the barred quantities are constants. The partial derivatives in Eq.(8) with respect to these new parameters are unchanged so that

$$\frac{\partial r_2}{\partial \tilde{r}_1} = 2\tilde{r}_1 + 2\overline{r}_1 + c, \ \ \frac{\partial r_2}{\partial \tilde{c}_2} = -1, \ \ \frac{\partial r_2}{\partial \tilde{c}_3} = 0 \ \ , \tag{19}$$

which on integration yields

$$\begin{aligned} r_2(\tilde{r}_1, \tilde{c}_2) &= \tilde{r}_1^2 + 2\overline{r}_1 \tilde{r}_1 + c\tilde{r}_1 + \tilde{X}_2 - \tilde{c}_2 \\ \vdots \qquad & \qquad \vdots \end{aligned} \tag{20}$$

with general structure

$$r_n(\tilde{r}_1, \tilde{c}_2, \ldots, \tilde{c}_n) = \hat{r}_n(\tilde{r}_1, \tilde{c}_2, \ldots, \tilde{c}_{n-1}) + \tilde{X}_n - \tilde{c}_n/(n-1) \ . \tag{21}$$

The $\tilde{X}_n$ are again constants of integration which are unknown unless a complete N^{n}LO calculation has been performed. If one applies the same rationale as before, where all RG-predictable terms are to be resummed, one finds , analogous to Eq.(18),

$$\mathcal{R}(Q) = \overline{a} + \overline{r}_1 \overline{a}^2 + \tilde{X}_2 \overline{a}^3 + \ldots + \tilde{X}_n \overline{a}^{n+1} + \ldots \ , \tag{22}$$

with $\overline{a}\equiv a(r_1=\overline{r}_1, c_2=\overline{c}_2, \ldots, c_n=\overline{c}_n, \ldots)$, which is just the perturbation series in the scheme with $r_1=\overline{r}_1, c_2=\overline{c}_2, \ldots, c_n=\overline{c}_n, \ldots$, or equivalently with $\tilde{r}_1=0, \tilde{c}_2=0, \ldots, \tilde{c}_n=0, \ldots$.

Thus by itself the principle of resumming all the RG-predictable terms at any given order of Feynman diagram calculation does not abolish the scheme dependence problem since the subset of RG-predictable terms depends on how the scheme dependence is parametrized, the parametrization ambiguity being precisely equivalent to the scheme dependence ambiguity. However, as we shall now argue, the parameter r_1 has a special status, being connected with the dimensionful renormalization scale μ and the physical energy scale Q on which $\mathcal{R}(Q)$ depends. Thus from Eq.(6) we see that we may write

$$r_1(\mu) = \left(b \ln \frac{\mu}{\tilde{\Lambda}_{\overline{MS}}} - b \ln \frac{Q}{\Lambda_{\mathcal{R}}} \right) , \tag{23}$$

with μ taken to be the $\overline{MS}$ scale as is customary. To simplify the discussion let us temporarily set $c=0$ and work in a scheme with $c_2=c_3=\ldots=c_n=\ldots=0$. Then the coupling $a(\mu)$ is given by

$$a(\mu) = 1/b \ln(\mu/\tilde{\Lambda}_{\overline{MS}}) , \tag{24}$$

and the NLO RG-improvement in Eq.(14) becomes a geometrical progression in r_1,

$$\mathcal{R}(Q) \approx a(\mu) + r_1(\mu) a^2(\mu) + r_1^2(\mu) a^3(\mu) + \ldots + r_1^n a^{n+1}(\mu) + \ldots . \tag{25}$$

Substituting Eq.(23) for $r_1(\mu)$, summing the geometrical progression and using Eq.(24) for $a(\mu)$ yields

$$\mathcal{R}(Q) \approx a(\mu) / \left[1 - \left(b \ln \frac{\mu}{\tilde{\Lambda}_{\overline{MS}}} - b \ln \frac{Q}{\Lambda_{\mathcal{R}}} \right) a(\mu) \right] = 1/b \ln(Q/\Lambda_{\mathcal{R}}) . \tag{26}$$

We see explicitly the cancellation of the unphysical $\ln(\mu/\tilde{\Lambda}_{\overline{MS}})$ terms in $r_1(\mu)$ with those in $a(\mu)$ to generate the correct physical Q-dependence of $\mathcal{R}(Q)$,

$$\mathcal{R}(Q) \approx 1/b \ln(Q/\Lambda_{\mathcal{R}}) + O(1/b \ln(Q/\Lambda_{\mathcal{R}}))^3 . \tag{27}$$

In contrast with standard NLO RG-improvement one has

$$\mathcal{R}(Q) \approx a(\mu) + r_1(\mu) a^2(\mu) , \tag{28}$$

where $\mu = xQ$ is taken proportional to Q with x a dimensionless constant, and $x = 1$ the so-called "physical scale" is often favoured. The resulting Q-dependence is

$$\mathcal{R}(Q)\approx 1/b(\ln(xQ/\tilde{\Lambda}_{\overline{MS}}) + O(1/b\ln(xQ/\tilde{\Lambda}_{\overline{MS}}))^2 \,, \tag{29}$$

which is, of course, x-dependent. All x-dependence cancels and the physical Q-dependence of Eq.(27) results if the geometric progression of Eq.(25) is not truncated but is resummed to all-orders. Truncating the resummation can result in considerable error in the extraction of $\Lambda_{\overline{MS}}$ from comparisons of NLO perturbative results with experiment. For many e^+e^- jet observables one has $r = r_1^{\overline{MS}}(\mu = Q)\approx 10$ [4] and the truncation of a geometric progression with a common ratio $ra\approx 1/2$ (taking $a = a_{\overline{MS}}(\mu = M_Z)\approx 0.05$) can lead to a sizeable overestimate of the true $\Lambda_{\overline{MS}}$ unless fortuitously the, as yet unknown , NNLO and higher invariants $X_2, X_3, \ldots$ compensate.

The complete resummation of Eq.(25) is forced on one if the renormalization scale μ is kept *independent* of the physical energy scale Q (i.e. μ held constant) since then the $\ln(Q/\Lambda_{\mathcal{R}})$ UV logarithms are the only source of Q-dependence. Standard fixed-order NLO RG-improvement is then manifestly unsatisfactory since it does not satisfy asymptotic freedom, $\mathcal{R}(Q)\to 0$ as $Q\to\infty$. Instead

$$\mathcal{R}_{NLO}(Q) = a(\mu) + \left(b\ln\frac{\mu}{\tilde{\Lambda}_{\overline{MS}}} - b\ln\frac{Q}{\Lambda_{\mathcal{R}}}\right)a^2(\mu) \,, \tag{30}$$

tends to $-\infty$ as $Q\to\infty$ with μ fixed. Only if the complete resummation in Eq.(25) is carried out does one build the correct $1/\ln(Q/\Lambda_{\mathcal{R}})$ behaviour.

Notice that use of the parametrization with $\tilde{r}_1 = r_1 - \overline{r}_1$ and $\overline{r}_1 = b\ln(xQ/\tilde{\Lambda}_{\overline{MS}}) - b\ln(Q/\Lambda_{\mathcal{R}})$ is equivalent to standard NLO RG-improvement with $\mu = xQ$ and yields the x-dependent result in Eq.(29) when all the NLO RG-predictable $\tilde{r}_1$ terms are resummed. The resummation is then manifestly incomplete with respect to UV logarithms since the r_1a^2 term is split into $\tilde{r}_1a^2 + \overline{r}_1a^2$ with the $\overline{r}_1a^2$ term not summed. Furthermore the constants of integration $\tilde{X}_2, \tilde{X}_3, \ldots$, which are unknown and hence omitted at NLO now

contain $\ln(xQ/\tilde{\Lambda}_{\overline{MS}})$ and $\ln(Q/\Lambda_{\mathcal{R}})$ terms. For instance

$$\tilde{X}_2 = X_2 + \left(b\ln\frac{xQ}{\tilde{\Lambda}_{\overline{MS}}} - b\ln\frac{Q}{\Lambda_{\mathcal{R}}} \right)^2 . \tag{31}$$

If these terms are included in the resummation the x-dependence disappears order-by-order reproducing the complete resummation of Eq.(25) and giving the correct physical Q-dependence in Eq.(27). So the parameter r_1 has a special status and should not be translated if one wishes to completely resum *all* the UV logarithms and reconstruct the correct physical Q-dependence.

With realistic non-zero c the NLO complete RG-improvement (CORGI) of Eq.(14) sums to a_0 satisfying Eq.(15) . This may be written in closed form as

$$\begin{aligned} \mathcal{R}(Q) &\approx \frac{-1}{c[1+W(z(Q))]} \\ z(Q) &\equiv -\frac{1}{e}\Big(\frac{Q}{\Lambda_{\mathcal{R}}}\Big)^{-b/c} , \end{aligned} \tag{32}$$

where $W(z)$ is the Lambert W-function defined implicitly by $W(z)\exp(W(z)) = z$ [8]. This is of course equivalent to standard "incomplete" NLO RG-improvement with $\mu = e^{-r/b}Q$, the FAC (or EC) scale , as noted earlier. Whilst CORGI yields μ-independent results there is still a dependence on the other dimensionless scheme parameters $c_2, c_3, \ldots$. That is one should parametrize the RS-dependence using r_1 (i.e. $\overline{r}_1 = 0$), but there is no preference for any particular parametrization $\tilde{c}_2, \tilde{c}_3, \ldots$. In the effective charge approach of Grunberg [2] one chooses $\overline{c}_2, \overline{c}_3, \ldots, \overline{c}_n$ so that the $\tilde{X}_2, \tilde{X}_3, \ldots, \tilde{X}_n$ are zero at N^{n}LO, corresponding to $r_1 = 0, r_2 = 0, \ldots, r_n = 0$. The use of $c_2, c_3, \ldots, c_n$ as parameters corresponds to an 't Hooft scheme [10] $\overline{c}_2 = 0, \overline{c}_3 = 0, \ldots, \overline{c}_n = 0$, and is *a priori* equally reasonable [2]. At NLO one conventionally makes this choice in any case.

We finally note that inverting the relation of Eq.(32) leads to the property of asymptotic scaling [11]

$$\lim_{Q\to\infty} Q\mathcal{F}(\mathcal{R}(Q)) = \Lambda_{\mathcal{R}} , \tag{33}$$

where $\mathcal{F}(x)$ is the universal QCD scaling function

$$\mathcal{F}(x) \equiv e^{-1/bx}(1+1/cx)^{c/b} \; . \tag{34}$$

Exactly this property is used in lattice gauge theory calculations of lattice coupling to assess how close to the continuum limit of infinite inverse lattice spacing one is. By using the exact relation between $\Lambda_{\mathcal{R}}$ and the *universal* parameter $\Lambda_{\overline{MS}}$ in Eq.(16) one arrives at the *universal* asymptotic scaling relation

$$\lim_{Q\to\infty} Q\mathcal{F}(\mathcal{R}(Q))e^{-r/b}(2c/b)^{c/b} = \Lambda_{\overline{MS}} \; . \tag{35}$$

This relation can be used analogously to the lattice scaling relation to assess how close to asymptotia in Q one is at current energies, for QCD observables calculated to NLO. One simply inserts the data for $\mathcal{R}(Q)$, and the corresponding NLO corrections $r \equiv r_1^{\overline{MS}}(\mu = Q)$, into Eq.(35). The asymptotic prediction is that all the data should lie on a single horizontal straight line corresponding to $\Lambda_{\overline{MS}}$. Deviations from this unambiguously indicate the presence of sub-asymptotic effects , and enable the estimation of relative differences between the uncalculated NNLO invariants X_2 , and possible power corrections, for different observables. Observation of the scaling property at some approximate level provides a well-founded starting point for further analyses attempting to resum large logarithmic corrections for jet observables or predict power corrections. In particular the sub-asymptotic effects are contained in a non-universal factor $\mathcal{G}(\mathcal{R}(Q))$ which approaches unity as $Q\to\infty$, so that

$$Q\mathcal{F}(\mathcal{R}(Q))\mathcal{G}(\mathcal{R}(Q))e^{-r/b}(2c/b)^{c/b} = \Lambda_{\overline{MS}} \; , \tag{36}$$

with

$$\mathcal{G}(\mathcal{R}(Q)) = 1 - (X_2/b)\mathcal{R}(Q) + \ldots + (K_0/\mathcal{R}^2)\mathcal{R}^{-c/b}e^{-1/b\mathcal{R}} + \ldots \; . \tag{37}$$

The K_0 term represents a possible power correction, here taken to be $1/Q$. Direct fits of X_2 and K_0 to the Q-dependence of the data can be performed along the lines discussed in Ref.[6].

This is to be contrasted with standard experimental analyses [5] which attempt to assess the size of uncalculated higher-order corrections by variation of the chosen renormalization scale $\mu = xQ$. As we have attempted to indicate here such an approach can be misleading , and leads to no information on the size of the uncalculated higher-order invariants $X_2, X_3, \ldots$.

Acknowledgements

We would like to thank John Campbell and Nigel Glover for a productive collaboration and numerous discussions which were instrumental in developing the ideas reported here.

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